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NOTE

Solution of the Shallow Water Equations Using Hybrid Grids

1. INTRODUCTION

In recent years a lot of knowledge has been gained concerning grids and numerical methods suitable for the numerical simulation of fluids and atmospheric modelling, in particular. Among the considerations leading to the choice of a numerical scheme for a particular model are accuracy, the conservation of integral invariants, numerical economy, ease of implementation, and a favorable behavior with respect to nonlinear instability.

Accurate schemes are easily obtained by using a high order of approximation. While many models use second-order numerical methods, the fourth-order schemes described by Kreiss *et al.* [1] are much more accurate for solutions which are smooth enough relative to the grid length. In the present paper we will use fourth-order schemes derived by finite elements, according to Cullen [33]. A model using this discretisation has been described by Carson *et al.* [2] and a survey of the finite element method was given by Navon [3]. Navon [4, 5], Staniforth *et al.* [6], Steppeler [8, 9], and Steppeler *et al.* [10] use atmospheric models based on the finite element method. The accuracy of this method for linear advection problems has been analysed by Gresho *et al.* [7] and it was found to be even greater than that of the fourth-order finite difference schemes of the Kreiss *et al.* [1] type. Finite difference methods are based on truncated Taylor series expansions, and with these methods it is therefore necessary to have a smooth solution in relation to the grid used. Such smoothness is often maintained by a numerical diffusion, which is present for numerical reasons. This is opposed to physical diffusion, which represents a physical process. In fact, most methods would give reasonable accurate results if this smoothness condition would be maintained by introducing a strong enough numerical diffusion into the model. However, a smooth solution means that many gridpoints are needed in order to describe an atmospheric structure. To use many gridpoints in this way is expensive in terms of computer resources, and therefore it is considered desirable to have as much structure in the model as possible, which means to minimize the amount of smoothing.

For this reason the behavior of numerical methods at the limit of the resolution is interesting, and it has turned out that bad accuracy for the shortest resolved scales can lead to noisy fields and requires a lot of extra diffusion. A high order of approximation will not necessarily solve such problems, as it improves, by definition, only the accuracy of the well-resolved scales.

Such problems of noise generation by inaccuracies in the poorly resolved scales do not concern the approximation of the advection terms of the dynamic equation. It is for these terms that the analysis of Gresho *et al.* [7] and Neta *et al.* [11] holds. Linear finite elements achieve a high accuracy when used on regular grids on unstaggered grids. The analysis of the advection term shows that it is adequately solved on an unstaggered Arakawa A-grid. In fact, the finite element scheme for advection can most conveniently be formulated on this grid. The use of vorticity and divergence as model variables [6, 12] is equivalent to grid staggering. This approach achieves only second-order approximation for the advection terms. Unstaggered linear finite elements for the advection term, as given in [2, 4], are much simpler to implement and will easily lead to a fourth-order approximation.

Early implementations of the finite element scheme [2] used the unstaggered Arakawa A-grid also for the terms of the dynamic equation responsible for the gravitational waves. In the light of later research by Schoenstadt [13, 14] and Williams [15], this cannot be considered as a good approach.

The approximation of the gravitational terms needs some kind of staggered grid in order to avoid severe noise problems. This request of staggered grids for the gravitational terms is widely accepted [16] and is connected to established mathematical principles [19]. Further explanations are given by Sani *et al.* [17, 18].

A practical way of incorporating the concept of staggering with two-dimensional finite element schemes was introduced by Steppeler [20]. A mixture of piecewise constant and linear basis functions is used, as first proposed for one-dimensional grids by Williams *et al.* [21].

A number of grids were investigated in [20], and the representation in the Arakawa C-grid turned out to be most suitable for the approximation of the gravitational terms. The numerical methods investigated in [20] are rather simple, and models based on this method should run very

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economically on computers. Since the staggering with this method is obtained by the use of low order basis functions, there is the possibility of a loss of accuracy.

Unfortunately, while the C-grid method is adequate for the approximation of the gravitational terms, it turned out [22] that, for the advection terms, the accuracy falls back to second order and that the essential advantage of linear finite elements [7] is lost. In order to avoid this difficulty and to make the practical use of staggered grids possible, the use of hybrid grids has been proposed by Steppeler [22]. This means the use of staggered and unstaggered grids in the same model, and the gravitational and advection terms can be computed in the grid which suits them most. The forward transformation from the unstaggered to the staggered grid is done by a simple interpolation. The backward transformation from the staggered to the unstaggered grid has to be the inverse of the former operation in order to recover the original field completely. If an interpolation were used for both forward and backward transformations, a smoothing would result from applications of the forward and backward operators. The use of two equivalent field representations is well known from the spectral method, which uses the spectral and gridpoint representations and, for the different terms of the dynamic equations, uses that representation which is most suitable.

This scheme was introduced and demonstrated in [22] for the one-dimensional barotropic equations, and the purpose of the present paper is to investigate the practical use of this method for two space dimensions. The staggered part of the hybrid scheme is the C-grid scheme introduced in [20].

Hybrid grids are introduced here to switch between an unstaggered high-order finite element approximation for the advection terms and a staggered treatment of the gravitational terms. However, this method can be used for many other purposes, whenever one wants to approximate a term on a grid different from the others.

Another field of application for hybrid grids would be the in use of semi-Lagrangian methods for the advection term. This has been introduced in [23–25] in order to obtain a more accurate solution. Recently, this method has drawn much interest as a means to achieve computational economy [26–28]. It is much easier applied on unstaggered grids, since the additional computational cost of a staggered grid may offset some of the computational economy achieved. Hybrid grids offer the possibility of using the A-grid for semi-Lagrangian advection and still incorporate the advantages of staggering for the gravitational terms. The method will be described in Section 2, and in Section 3 a numerical example will be given.

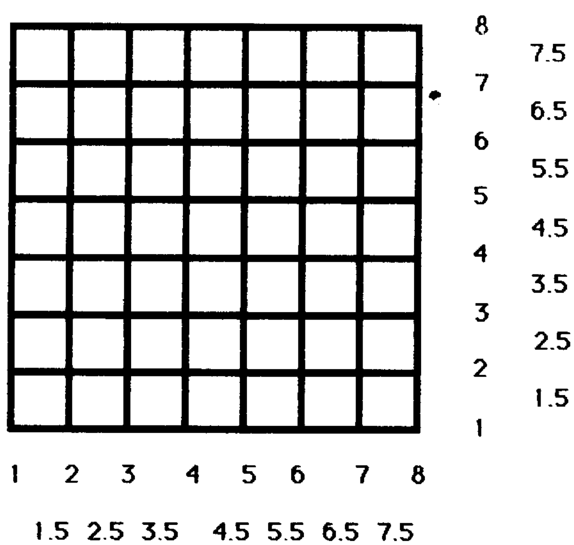

FIG. 1. The 8 by 8 grid used for the calculations.

2. DESCRIPTION OF THE NUMERICAL METHOD

We solve the shallow water equations on an f-plane:

$$\begin{aligned}\dot{U} &= -UU_x - VU_y + fV - H_x\\ \dot{V} &= -UV_x - VV_y - fU - H_y\\ \dot{H} &= -(U(H-H_0))_x - (V(H-H_0))_y.\end{aligned} \tag{1}$$

In Eq. (1) U, V are the velocity components, $H = g\phi$, with g being the gravitational constant and ϕ the height field, $H_0 = g\phi_0$, with ϕ_0 being the bottom topography. X and Y are the space coordinates. We solve Eq. (1) on an 8 by 8 grid, which is shown in Fig. 1. This is a rather coarse resolution, as test calculations by Grammeltvedt [29] indicated that at least 12 gridpoints on a wave are necessary to obtain a good simulation of an atmospheric wave. However, since we intend to use the highly accurate finite element method [7] for the advection terms, we expect this grid to provide sufficient resolution for a test calculation. Periodic boundary conditions are used in X-direction. This is implemented by identifying rows 1 and 8 in Fig. 1. The scheme using the C-grid as basic representation is numerically equivalent. At the top and bottom solid wall boundaries $V = 0$ and $Hy = Uy = 0$ are assumed.

The main nodepoint grid in Fig. 1 is indicated by integer indices. To define the staggered grid representations, half integers are used in addition, which define locations between two main nodepoints. The discretisation is done on grid squares, and Fig. 2 shows the two field representations used here. The A-grid uses the main nodepoint grid for all fields U, V, H. The field in the C-grid of Fig. 2 is obtained by a transformation from the main nodepoint representation. The time derivative is then transformed back to the A-grid representation.

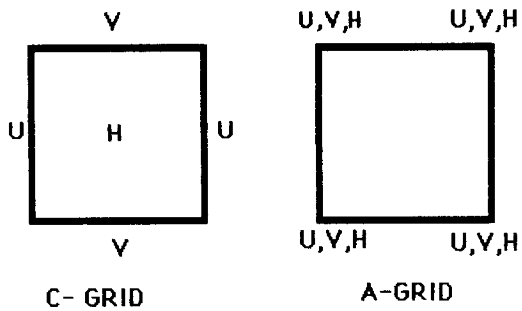

FIG. 2. Field representation in the staggered Arakawa C-grid and the unstaggered A-grid.

The hybrid grid treatment splits the dynamic equation (1) in the following way:

$$\dot{U} = \dot{A}^U + \dot{G}^U$$
$$\dot{V} = \dot{A}^V + \dot{G}^V \quad (2)$$
$$\dot{H} = \dot{A}^H + \dot{G}^H.$$

The advection part A and the gravitational part G is defined in the following way:

$$\begin{aligned} \dot{A}^U &= -UU_x - VU_y & \dot{G}^U &= +fV - H_x \\ \dot{A}^V &= -UV_x - VV_y & \dot{G}^V &= -fU - H_y \\ \dot{A}^H &= -UH_x - VH_y & \dot{G}^H &= (-U_x - V_y)(H - H_0). \end{aligned} \quad (3)$$

The advection terms A are approximated directly in the main nodepoint grid. The discretisation used here is described in Steppeler *et al.* [10]. The method referenced in [10] as the standard Galerkin scheme is used. Any other method to which the reader may be familiar, as, for example, a semi-Lagrangian scheme, can be used in a hybrid grid combined with a staggered treatment of the gravitational terms G. The gravitational terms G are treated according to the C-grid scheme defined by Steppeler [20]. This scheme defines fields with half integer indices by piecewise constant elements and fields with integer indices by piecewise linear elements and uses standard Galerkin approximations. The C-grid representation is obtained from the A-representation. After the G-terms are calculated, the result is transformed back into the A-grid. The C-grid representation is obtained from the A-grid by the transformations

$$\begin{aligned} U_{i,j+1/2} &= \tfrac{1}{2}(U_{i,j} + U_{i,j+1}) \\ V_{i+1/2,j} &= \tfrac{1}{2}(V_{i,j} + V_{i+1,j}) \\ H_{i+1/2,j+1/2} &= \tfrac{1}{4}(H_{i,j} + H_{i+1,j} + H_{i,j+1} + H_{i+1,j+1}). \end{aligned} \quad (4)$$

The backward transformation from the C-grid to the A-grid is obtained by solving Eq. (4) for the amplitudes $U_{i,j}$, $V_{i,j}$, $H_{i,j}$. For V and H this is easily done using Gaussian elimination (see [30]). For the U-field there is the difficulty that for every i there are eight j-amplitudes, but only seven half-lavel amplitudes $U_{i,j+1/2}$. In order to close Eqs. (4), we demand that the two ΔX-wave has the minimum amplitude

$$\sum_{i=1}^{8} u_{i,j}(-1)^i = 0. \quad (5)$$

Alternative closure conditions, like minimizing the least square second field derivatives, have been used successfully. The fact that all fields are defined at the same locations makes the scheme rather easy to implement and it requires small computer resources.

3. NUMERICAL RESULTS

The initial values are the same as used in [8, 10, 29]. Plots of the initial fields are given in [8],

$$H(X, Y) = H_0 + H_1 \tan\left(\frac{9(Y - Y_0)}{2D}\right) + H_2\left(1 \Big/ \cosh^2\left(\frac{9(Y - Y_0)}{D}\right)\right) \quad (6)$$

with

$$\begin{aligned} H_0 &= 20{,}000\ \mathrm{m^2 s^{-2}} \\ H_1 &= 4400\ \mathrm{m^2 s^{-2}} \\ H_2 &= 2660\ \mathrm{m^2 s^{-2}} \\ L &= D = 400 \\ \Delta X &= \frac{40000\ 11}{7}\ 10^5\ \mathrm{m} \\ f &= 0.0001 \\ \Delta t &= 900\ \mathrm{s}. \end{aligned}$$

The leapfrog time discretisation is used. In order to demonstrate the effect of grid staggering on the solution, a strong mountain forcing was introduced by using a mountain barrier situated at column 6. An analytic solution of the one-dimensional problem was obtained by Edelmann [31], and this suggests that a stationary ridge will form over the mountain. The flow, as generated by the initial values (6), is rather unbalanced and creates fast moving gravitational waves. In order to obtain a balanced solution, the forecast of the first 12 h is time averaged and after this a 48-h forecast is done. No time diffusion was necessary to control the computational mode of the leapfrog scheme. When in semi-implicit and semi-Lagrange versions of the model the time step was substantially increased; a time filter according to Asselin [34] became necessary.

The resulting H field is shown in Figs. 3a,b. The forecast shown in Fig. 3a uses the unstaggered A-grid for both the

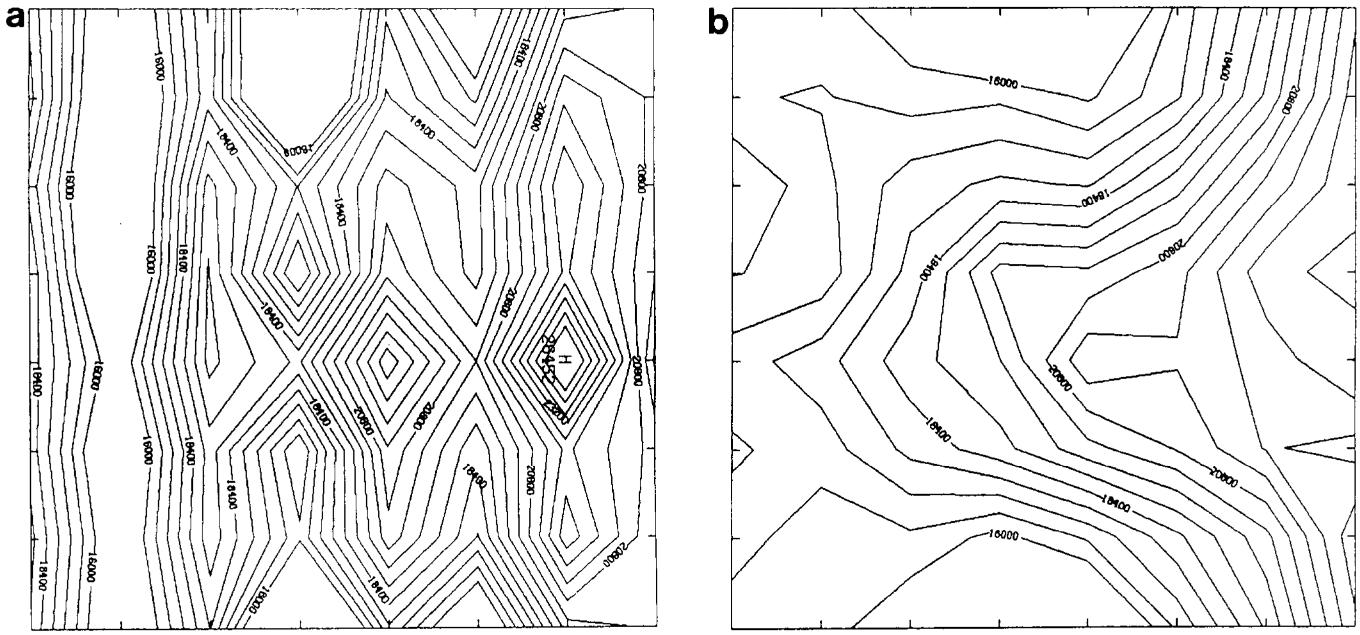

FIG. 3. (a) H-field after a 24 h prediction using the unstaggered A-grid for all terms of the dynamic equation. (b) as (a), but using a hybrid grid which computes the gravitational terms on the C-grid.

advection and the gravitational terms. The solution is contaminated by very strong noise features. According to arguments given in [13, 14, 19, 20], this was to be expected, since A-grid approximations of the gravitational terms do not allow for a proper geostrophic adjustment. Figure 3b shows the result using the hybrid grid, combining an A-grid scheme [10] for the advection terms with a C-grid scheme [20] for the gravitational terms. The noise features are virtually absent in this solution, and this scheme can therefore be expected to need much less diffusion than A-grid schemes. Figure 3 shows that the hybrid grid scheme, which uses the grid staggering for the gravitational terms only, shows the same behavior in practice as is expected from staggered schemes. This confirms theoretical expectations [15, 20]. While the hybrid scheme has been shown to demonstrate the low noise features of C-grid schemes, its representation of the advection is much more accurate. The C-grid representation of advection according to [21] is equivalent to the second-order centered difference approximation. The A-grid advection using linear finite elements, which is used with the hybrid grid here, is fourth-order accurate. An investigation of the accuracy of advection of these two schemes has been performed in [33, 7], and the results were very much in favour of the linear finite elements, which here are used for advection with the hybrid grid.

4. CONCLUSIONS

The practical viability of hybrid grids, which use grid staggering for the gravitational terms only and keep an unstaggered grid for the advection terms, has been investigated. This offers the possibility of applying the low-order staggered finite element schemes [20] together with a high-order treatment of the advection terms. The numerical behavior of the solution is such as expected in a staggered grid.

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